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Nguyen Thang DAO, Julio DAVILA

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Implementing steady state efficiency in overlapping generations economies with environmental externalities

Nguyen Thang DAO and Julio DÁVILA

Abstract

We consider in this paper overlapping generations economies with pollution resulting from both consumption and production. The competitive equilibrium steady state is compared to the optimal steady state from the social planner's viewpoint. We show that any competitive equilibrium steady state whose capital-labor ratio exceeds the golden rule ratio is dynamically inefficient. Moreover, the range of dynamically efficient steady states capital ratios increases with the effectiveness of the environment maintainance technology, and decreases for more polluting production technologies. We characterize some tax and transfer policies that decentralize as a competitive equilibrium outcome the social planner's steady state.

Resumé

On étudie dans ce papier des économies de générations imbriquées avec pollution provenant aussi bien de la consommation que de la production. L'état stationnaire d'équilibre concurrentiel est compare à l'état stationnaire optimal du point de vue du planificateur. On montre que lorsque le ratio capital-travail excède à l'état stationnaire de l'équilibre concurrentiel celui de la règle d'or, alors le premier est dynamiquement inefficace. De plus, l'intervalle d'états stationnaires dynamiquement efficaces s'accroît avec l'effectivité de la technologie de maintient de l'environnement, et diminue lorsque la technologie de production est plus polluante. On caractérise des politiques fiscales qui décentralisent l'état stationnaire du planificateur comme équilibre concurrentiel.

Keywords: overlapping generations, environmental externality, tax and transfer policy.

Mots clés: générations imbriquées, externalités environementales, politiques fiscales.

JEL Classification: D62, E21, H21, H41

1 Introduction

Environmental externalities in economies with overlapping generations have been studied since at least the 1990s. In particular, the effects of environmental externalities on dynamic inefficiency, productivity, health and longevity of agents have been addressed, as well, as the policy interventions that may be needed. While in most papers pollution is assumed to come from production, and the environment is supposed to improve or degrade by itself at a constant rate (Marini and Scaramozzino 1995; Jouvet et al 2000; Jouvet, Pestieau and Ponthiere 2007; Pautrel 2007; Gutiérrez 2008), other papers assume that pollution comes from consumption (John and Pecchenino 1994; John et al. 1995; Ono 1996). As a consequence of the differing assumptions, the effect of environmental externalities on capital accumulation vary widely across papers. Specifically, John et al. (1995) showed that when only consumption pollutes, the economy accumulates less capital than that what would be optimal. Conversely, Gutiérrez (2008) showed that when only production pollutes, the economy accumulates instead more capital than the optimal level. This is so because in John et al. (1995) agents pay taxes to maintain environment when young, so that an increased pollution reduces their savings; however, in Gutiérrez (2008) pollution increases health costs in old age, leading agents to save more to pay for them. The difference seems therefore to come from when the taxes are paid (when young or old) rather than from whether pollution comes from production or consumption. Another main difference between John et al. (1995) and Gutiérrez (2008) is their different assumptions about the ability of environment to recover from pollution. John et al. (1995) assumes that environment naturally degrades over time, while Gutiérrez (2008) assumes that environment recovers naturally.

This paper aims at disentangling the effects of both production and consumption on environment. Specifically, as in John et al. (1994, 1995), we assume that the environment degrades naturally over time at a constant rate and that young agents devote part of their income to maintain it.[1] In this setup, we characterize the range

[1]In John et al. (1994, 1995), only the consumption of old agents pollutes, young agents do not consume. In Ono (1996), it is assumed that consumption of both young and old agents degrade the environment but with a period lag. Here, we assume also that consumptions of both old and young agents and production pollute without decay.

of dynamically inefficient capital-labor ratios. Next, we introduce taxes and transfer policies that decentralize the first-best steady state as a competitive equilibrium steady state.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 characterizes its competitive equilibria. Section 4 presents the problem of the social planner, defines the efficient allocation with and without discounting, and characterizes the range of dynamically inefficient capital ratios (Proposition 1). The competitive equilibrium steady state and the planner's steady state are compared in Section 5, where we introduce some tax and transfer schemes that decentralize the planner's steady state as market outcome (from Proposition 2 to Proposition 9). Section 6 concludes the paper.

2 The model

We consider the overlapping generations economy in Diamond (1965) with a constant population of identical agents. The size of each generation is normalized to one. Each agent lives two periods, say young and old. When young, an agent is endowed with one unit of labor which he supplies inelastically. Agents born in period t divide their wage w_t between consumption when young c_0^t, investment in maintaining the environment m^t, and savings k^t lent to firms to be used in $t+1$ as capital for a return rate r_{t+1}. The return of savings $r_{t+1}k^t$ is used up as old age consumption. Agents born at date t have preferences over their consumptions when young and old $(c_0^t, c_1^t) \in \mathbb{R}_+^2$ and the environmental quality when old, $E_{t+1} \in \mathbb{R}$, represented by $u(c_0^t) + v(c_1^t) + \phi(E_{t+1})$ with u', v', $\phi' > 0$, u'', v'', $\phi'' < 0$, and $u'(0) = v'(0) = +\infty$, $u'(+\infty) = v'(+\infty) = 0$.

Environmental quality evolves according to

$$E_{t+1} = (1-b)E_t - \alpha F(K_{t+1}, L_{t+1}) - \beta(c_0^{t+1} + c_1^t) + \gamma m^t$$

for some $\alpha, \beta, \gamma > 0$ and $b \in (0, 1]$, where F is a Cobb-Douglas production function $F(K_t, L_t) = AK_t^{\theta}L_t^{1-\theta}$. Capital fully depreciates in each period. Under perfect competition, the representative firm maximizes profits solving

$$\underset{K_t, L_t \geq 0}{Max} F(K_t, L_t) - r_t K_t - w_t L_t$$

so that the wage rate and the rental rate of capital are, in each period t, the marginal productivity of labor and capital respectively. Since population is normalized to 1, period t aggregate savings (i.e. period $t+1$ aggregate capital K_{t+1}) and labor supply are k_t and 1 respectively, and the wage and rental rate of capital faced by the agent born at period t are

$$r_{t+1} = F_K(k^t, 1) = \theta A (k^t)^{\theta-1} \tag{1}$$

$$w_t = F_L(k^{t-1}, 1) = (1-\theta) A (k^{t-1})^{\theta} \tag{2}$$

Environmental quality converges autonomously to a natural level normalized to zero at a rate b that measures the speed of reversion to this level. Nonetheless, production and consumption degrade environmental quality by an amount $\alpha F(K_{t+1}, 1)$ and $\beta(c_0^{t+1} + c_1^t)$ respectively, while young agents can improve the environmental quality by an amount γm^t if they devote a portion m^t of their income to that end.[2]

The life-time utility maximization problem of the representative agent is

$$\underset{\substack{c_0^t, c_1^t, k^t, m^t \geq 0 \\ E_t,\, E_{t+1}^e}}{Max} u(c_0^t) + v(c_1^t) + \phi(E_{t+1}^e) \tag{3}$$

subject to

$$c_0^t + k^t + m^t = w_t \tag{4}$$

$$c_1^t = r_{t+1} k^t \tag{5}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1}, 1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{6}$$

[2]One can thus interpret environmental quality as including any characteristic that make the environment more apt for human life, like the cleanness of rivers and atmosphere, and the quality of soil or groundwater, etc. It also includes the state of forests, agricultural lands, parks and gardens, which left on their own naturally revert to wilderness, unless subjeCt to regular maintenance.

$$E^e_{t+1} = (1-b)E_t - \alpha F(K_{t+1}, 1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{7}$$

given E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, w_t, r_{t+1} as well as the expected consumption of the next generation young agent $c_0^{t+1,e}$. Since the presentative agent is assumed to be negligible within his own generation, he thinks of the impact of his savings k^t on aggregate capital K_{t+1} to be negligible as well, ignoring that actually $K_{t+1} = k^t$ at equilibrium. This assumption implies that he does not internalize the impact of the savings decision on environment via production.

Agent t's optimal choice $(c_0^t, c_1^t, k^t, m^t, E_t, E^e_{t+1})$ is therefore characterized by the first-order conditions

$$u'(c_0^t) - [\beta(1-b) + \gamma]\, \phi'(E^e_{t+1}) = 0 \tag{8}$$

$$v'(c_1^t) - \left[\beta + \frac{\gamma}{r_{t+1}}\right] \phi'(E^e_{t+1}) = 0 \tag{9}$$

$$c_0^t + k^t + m^t - w_t = 0 \tag{10}$$

$$c_1^t - r_{t+1}k^t = 0 \tag{11}$$

$$E_t - (1-b)E_{t-1} + \alpha F(k^{t-1}, 1) + \beta(c_0^t + c_1^{t-1}) - \gamma m^{t-1} = 0 \tag{12}$$

$$E^e_{t+1} - (1-b)E_t + \alpha F(K_{t+1}, 1) + \beta(c_0^{t+1,e} + c_1^t) - \gamma m^t = 0 \tag{13}$$

as an implicit function of E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, w_t, r_{t+1} and $c_0^{t+1,e}$ as long as the Jacobian matrix of the left-hand-side of the system above with respect to c_0^t, c_1^t, k^t, m^t, E_t, E^e_{t+1} is regular at the solution. The existence of the optimal solution is established in Appendix A1 and the regularity of the Jacobian matrix at equilibrium is established in Appendix A2. For these FOCs to be not only necessary but also sufficient for the solution to be a maximum, the second order conditions (SOCs) are shown to hold at equilibrium in Appendix A3.

3 Competitive equilibria

Perfect foresight competitive equilibria are characterized by (i) the agent's utility maximization under the budget constraints, with correct expectations, (ii) the firms' profit maximization determining factors' prices, and (iii) the dynamics of environment. Therefore, a competitive equilibrium allocation $\{c_0^t,\ c_1^t,\ k^t,\ m^t,\ E_{t+1}\}_t$ is solution to the system of equations

$$u'(c_0^t) - [\beta(1-b)+\gamma]\,\phi'(E_{t+1}) = 0 \tag{14}$$

$$v'(c_1^t) - \left[\beta + \frac{\gamma}{F_K(k^t,1)}\right]\phi'(E_{t+1}) = 0 \tag{15}$$

$$c_0^t + k^t + m^t - F_L(k^{t-1},1) = 0 \tag{16}$$

$$c_1^t - F_K(k^t,1)k^t = 0 \tag{17}$$

$$E_{t+1} - (1-b)E_t + \alpha F(k^t,1) + \beta(c_0^{t+1} + c_1^t) - \gamma m^t = 0 \tag{18}$$

Note that the feasibility of the allocation of resources is guaranteed by the agent's budget constraints (16) and (17), since at t

$$c_0^t + c_1^{t-1} + k^t + m^t = F_K(k^{t-1},1)k^{t-1} + F_L(k^{t-1},1) = F(k^{t-1},1)$$

A perfect foresight competitive equilibrium steady state, in particular, is a $(c_0,\ c_1,\ k,\ m,\ E)$ solution to the system of equations

$$u'(c_0) - [\beta(1-b)+\gamma]\,\phi'(E) = 0$$

$$v'(c_1) - \left[\beta + \frac{\gamma}{F_K(k,1)}\right]\phi'(E) = 0$$

$$c_0 + k + m - F_L(k,1) = 0$$

$$c_1 - F_K(k,1)k = 0$$

$$bE + \alpha F(k,1) + \beta(c_0 + c_1) - \gamma m = 0$$

The perfect foresight competitive equilibria of this economy follow a dynamics represented by a first-order difference equation, because of the regularity of the associated Jacobian matrix of the left hand side of the system of equations above with respect to c_0^{t+1}, c_1^t, k^t, m^t, E_{t+1} (see Appendix A1).

4 The social planner's choice with and without discounting

In this section, we consider the optimal allocation from the viewpoint of a social planner that allocates resources in order to maximize a weighted sum of the welfare of all current and future generations. The allocation selected by the social planner, which is optimal in the Pareto sense, is a solution to the problem

$$\underset{\left\{c_0^t, c_1^t, k^t, m^t, E_{t+1}\right\}_{t=0}^{\infty}}{Max} \sum_{t=0}^{\infty} \frac{1}{(1+R)^t} \left[u(c_0^t) + v(c_1^t) + \phi(E_{t+1})\right] \tag{19}$$

subject to, $\forall t = 0, 1, 2, ...,$

$$c_0^t + c_1^{t-1} + k^t + m^t = F(k^{t-1}, 1) \tag{20}$$

$$E_{t+1} = (1-b)E_t - \alpha F(k^{t-1}, 1) - \beta(c_0^{t+1} + c_1^t) + \gamma m^t \tag{21}$$

given some initial conditions c_1^{-1}, k^{-1}, E_0, where $0 \leq R$ is the social planner's subjective discount rate.[3] The first constraint (20) of the problem is the resource constraint of the economy in period t requiring that the total output in that period is split into consumptions of the current young and old, savings for next period's capital, and environmental maintenance. The second constraint (21) is the dynamics of the environmental quality.

The social planner's choice of a steady state is a $(\bar{c}_0, \bar{c}_1, \bar{m}, \bar{k}, \bar{E})$ satisfying (see Appendix A4)

[3] The discount rate R is strictly positive when the social planner cares less about a generation's welfare the further away in the future that generation is, while R equals to zero when she cares about all generations equally, no matter how far in the fuure they may be.

$$u'(\bar{c}_0) = (1+R)\frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E}) \tag{22}$$

$$v'(\bar{c}_1) = \frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E}) \tag{23}$$

$$F_K(\bar{k}, 1) = \frac{\gamma(1+R)}{\gamma - (1+R)\alpha} \tag{24}$$

$$\bar{c}_0 + \bar{c}_1 + \bar{k} + \bar{m} = F(\bar{k}, 1) \tag{25}$$

$$b\bar{E} + \alpha F(\bar{k}, 1) + \beta(\bar{c}_0 + \bar{c}_1) - \gamma\bar{m} = 0 \tag{26}$$

(the planner's discount rate R cannot be arbitrarily high for the optimal steady state to be characterized as above, specifically $\gamma > (1+R)\alpha$ needs to hold, which requires $\gamma > \alpha$, so that $F_K(\bar{k}, 1) > 0$). More specifically, in the case of the social planner caring about all generations equally, i.e. $R = 0$, the planer's steady state is the so-called golden rule steady state $\{c_0^*, c_1^*, k^*, m^*, E^*\}$ that maximizes the utility of the representative agent and is characterized by being a solution to the system

$$u'(c_0^*) = \frac{\gamma + \beta}{b}\phi'(E^*) \tag{27}$$

$$v'(c_1^*) = \frac{\gamma + \beta}{b}\phi'(E^*) \tag{28}$$

$$F_K(k^*, 1) = \frac{\gamma}{\gamma - \alpha} \tag{29}$$

$$c_0^* + c_1^* + k^* + m^* = F(k^*, 1) \tag{30}$$

$$bE^* + \alpha F(k^*, 1) + \beta(c_0^* + c_1^*) - \gamma m^* = 0 \tag{31}$$

Note that, from (27) and (28), the marginal utility of consumption of the young agent must equal that of the consumption of the old agent.

Diamond (1965) shows that in the standard OLG model without pollution externalities, a competitive equilibrium steady state whose capital per worker exceeds the golden rule level is dynamically inefficient. Notwithstanding, Gutiérrez (2008) shows that, when the pollution externality is large enough, there are dynamically efficient competitive equilibrium steady state capital ratios that exceed the golden rule capital ratio. Specifically, Gutiérrez (2008) shows the existence of a "super golden rule" level of capital ratio, beyond the golden rule level, such that any economy with pollution externalities whose stationary capital ratio exceeds this level is necessarily dynamically inefficient. Nonetheless, it should be noted that in Gutiérrez (2008) (i) pollution externalities only from production are taken into account; (ii) the environment recovers itself overtime at a constant rate; (iii) no resource is devoted to maintaining the environment; and (iv) the pollution externality decreases the utility of the agents only indirectly by requiring each agent to pay for extra health costs in the old age. In this paper, we consider instead an economy with pollution externalities coming from both production and consumption, in which the environment degrades itself over time, and the quality of the environment can be improved through maintenance. Also the quality of environment directly affects the utility of the agents. As a consequence, this paper shows instead that, as in Diamond (1965) and in contrast with Gutiérrez (2008), in an economy with consumption and production pollution externalities, the golden rule capital ratio is still the highest level of capital ratio that is dynamically efficient.

Proposition 1: *In a Diamond (1965) overlapping generations economy with consumption and production pollution, for an efficient enough cleaning technology, compared to the marginal polluting impact of production (specifically, for $\gamma > \alpha$ in the model), the golden rule (i.e. the planner's steady state choice without discounting) is the highest dynamically efficient capital ratio.*

Proof: Since $F_{KK}(k,1) < 0$ for all k, the planner's optimal capital ratio $\bar{k}$ is implicitly defined to be a differentiable function $\bar{k}(R)$ of R by the condition

$$F_K(\bar{k},1) = \frac{\gamma(1+R)}{\gamma - (1+R)\alpha}$$

whose derivative, by the implicit function theorem, is

$$\bar{k}'(R) = \frac{1}{F_{KK}(\bar{k}(R),1)} \left(\frac{\gamma}{\gamma - (1+R)\alpha} \right)^2 < 0 \qquad (32)$$

So, $\bar{k}$ is decreasing in R. Hence, $\bar{k}(R)$ is maximal when $R = 0$, which is corresponds to the golden rule level of capital k^*■

Proposition 1 shows that any steady state capital ratio exceeding k^* is dynamically inefficient. From (29) the golden rule capital ratio k^* is decreasing in the production pollution parameter α. It is, however, increasing in the environment maintaining technology γ. Hence, the more polluting is production, the smaller the range of steady state allocations that are dynamically efficient for some discount factor R. Similarly, the more effective is the environment maintainance technology, the bigger the range of steady state allocations that are dynamically efficient for some discount factor R.

5 Policy implementation of the planner's optimal steady state

In this section, we provide tax and transfer policies allowing to implement the planner's optimal steady state. Ono (1996) and Gutiérrez (2008) introduced also tax and transfer schemes to decentralize the golden rule steady state in the context of the pollution externalities they consider (from consumption and production only, respectively). However, their schemes uphold the golden rule once the economy is already at that steady state. Nevertheless, in this section we provide policies that lead the economy towards the social planer's steady state and will keep it there once reached (for the golden rule, the social planner's discount rate just needs to be set to $R = 0$). The policies fulfill this in two stages. In the first stage, in the period t, taxes and transfers are set in order to make the agent born in period t choose his savings and consumption when old to be equal to the optimal steady state capital ratio and optimal steady state old agent's consumption, respectively, from the viewpoint of the social planner. Then in the second stage, taxes and transfers are reset to uphold the planner's steady state. The first scheme based on the taxation of consumption is presented next in detail. The subsequent schemes work analogously.

5.1. Taxes on consumptions

Suppose that, at the begining of period t, the social planner announces proportional taxes and lump-sum transfers. Letting τ_0^t and τ_1^t be the tax rate on agent t's consumption when young and old respectively, T_0^t a lump-sum tax (if positive) levied on agent t's income when young, and T_1^t a lump-sum transfer (if positive) to the same agent when old at date $t+1$, the problem of agent t is then

$$\underset{\substack{c_0^t, c_1^t, k^t, m^t \geq 0 \\ E_t, E_{t+1}^e}}{Max} u(c_0^t) + v(c_1^t) + \phi(E_{t+1}^e) \tag{33}$$

subject to

$$(1+\tau_0^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{34}$$

$$(1+\tau_1^t)c_1^t = r_{t+1}k^t + T_1^t \tag{35}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1}, 1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{36}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1}, 1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{37}$$

given c_1^{t-1}, k^{t-1}, m^{t-1}, E_{t-1}, w_t, r_{t+1} and $c_0^{t+1,e}$. Note again that in equation (37), the agent, being negligible within his generation, ignores the fact that $K_{t+1} = k^t$ and hence is unable to internalize the effect of the savings decisions on environment through the aggregate output. Hence, the first-order conditions characterizing agent t's optimal choice are

$$u'(c_0^t) = \left[\beta(1-b) + \gamma(1+\tau_0^t)\right] \phi'(E_{t+1}^e) \tag{38}$$

$$v'(c_1^t) = \left[\beta + \frac{\gamma(1+\tau_1^t)}{F_K(k^t, 1)}\right] \phi'(E_{t+1}^e) \tag{39}$$

$$(1+\tau_0^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{40}$$

$$(1+\tau_1^t)c_1^t = r_{t+1}k^t + T_1^t \tag{41}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{42}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{43}$$

At a perfect foresight equilibrium the wage rate and capital return are given by the labor and capital productivities respectively, and forecasts coincide with actual values, i.e. $E_{t+1}^e = E_{t+1}$ and $c_0^{t+1,e} = c_0^{t+1}$.

The next proposition shows that the tax rates and lump-sum transfers can be set at levels that make agent t choose the planner's steady state capital ratio and consumption when old.

Proposition 2: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, for any given period t, there exists a period by period budget balanced policy of consumption taxes, and lump-sum taxes and transfers, $(\tau_0^t, \tau_1^t, T_0^t, T_1^t)$, that implements at $t+1$ the planner's steady state capital ratio $\bar{k}$ and consumption when old $\bar{c}_1$ at a competitive equilibrium.*

Proof: Let the tax rates be

$$\tau_0 = \frac{[\gamma + \beta(1+R)](1+R) - [\gamma + \beta(1-b)](b+R)}{(b+R)\gamma} > 0$$

$$\tau_1 = \frac{1+R}{b+R} \cdot \frac{\gamma + \beta(1-b)}{\gamma - \alpha(1+R)} - 1 > 0$$

then equations (38), (39) become equations (22), (23) when the equation (24) characterizing the planner's steady state capital ratio, $F_K(\bar{k},1) = \frac{1+R}{1-(1+R)\alpha/\gamma}$, holds. Therefore agent t's chooses ice the planner's steady state capital ratio $\bar{k}$ and consumption when old $\bar{c}_1$ under the consumption tax rates τ_0, τ_1 above if, and only if,

$$u'(c_0^t) = (1+R)\frac{\gamma + \beta(1+R)}{b+R}\phi'(E_{t+1}) \tag{44}$$

$$v'(\bar{c}_1) = \frac{\gamma + \beta(1+R)}{b+R}\phi'(E_{t+1}) \tag{45}$$

$$(1+\tau_0)c_0^t + \bar{k} + m^t = F_L(k^{t-1},1) - T_0^t \tag{46}$$

$$(1+\tau_1)\bar{c}_1 = F_K(\bar{k},1)\bar{k} + T_1^t \tag{47}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_0^t + c_1^{t-1} + \tau_0 c_0^t + T_0^t) + \gamma m^{t-1} \tag{48}$$

$$E_{t+1} = (1-b)E_t - \alpha F(\bar{k},1) - \beta(c_0^{t+1} + \bar{c}_1) + \gamma m^t \tag{49}$$

given E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, $c_0^{t+1} = c_0^{t+1,e}$. Note that this is a system in $\{c_0^t, T_0^t, T_1^t, m^t, E_t, E_{t+1}\}$ (note also that in the equation (48) the consumption when old of agent $t-1$ is now $\tilde{c}_1^{t-1} = c_1^{t-1} + \tau_0 c_0^t + T_0^t$ since he receives as a lump-sum transfer the tax raised from the young agent t, $\tau_0 c_0^t + T_0^t$, so that the government's budget is balanced.

The solution in $\{c_0^t, T_0^t, T_1^t, m^t, E_t, E_{t+1}\}$ to the system of equations (44)-(49) is the following. Given that $\phi'' < 0$, from (45) and (23)

$$\left.\begin{array}{r} v'(\bar{c}_1) = \frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+1}) \\ v'(\bar{c}_1) = \frac{\gamma+\beta(1+R)}{b+R}\phi'(\bar{E}) \end{array}\right. \Rightarrow E_{t+1} = \bar{E}$$

then from (44), (22) and $E_{t+1} = \bar{E}$,

$$\left.\begin{array}{r} u'(c_0^t) = (1+R)\frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+1}) \\ u'(\bar{c}_0) = (1+R)\frac{\gamma+\beta(1+R)}{b+R}\phi'(\bar{E}) \\ E_{t+1} = \bar{E} \end{array}\right. \Rightarrow c_0^t = \bar{c}_0$$

Also from (47) it follows

$$T_1^t = (1+\tau_1)\bar{c}_1 - F_K(\bar{k},1)\bar{k}$$

and from (46) and $c_0^t = \bar{c}_0$ we have

$$m^t = F_L(k^{t-1},1) - (1+\tau_0)\bar{c}_0 - \bar{k} - T_0^t$$

It can also be easily checked that, substituting $E_{t+1} = \bar{E}$, $c_0^t = \bar{c}_0$, $m^t = F_L(k^{t-1},1) - (1+\tau_0)\bar{c}_0 - \bar{k} - T_0^t$ and equation (48) into equation (49) we have

$$\bar{E} = I - [\beta(1-b) + \gamma]\, T_0^t$$

where

$$I = (1-b)\left[(1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta c_1^{t-1} + \gamma m^{t-1}\right] - \alpha F(\bar{k},1) \\ - \left[\beta(1-b)+\gamma\right](1+\tau_0)\bar{c}_0 - \beta(c_0^{t+1} + \bar{c}_1) + \gamma\left[F_L(k^{t-1},1) - \bar{k}\right]$$

so that

$$T_0^t = \frac{I - \bar{E}}{\beta(1-b)+\gamma}$$

and hence, from equation (48),

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_1^{t-1} + (1+\tau_0)\bar{c}_0 + \frac{I-\bar{E}}{\beta(1-b)+\gamma}) + \gamma m^{t-1} \tag{50}$$

Therefore, the solution to the system of equations (44)-(49) is

$$\begin{pmatrix} c_0^t \\ T_0^t \\ T_1^t \\ m^t \\ E_t \\ E_{t+1} \end{pmatrix} = \begin{pmatrix} \bar{c}_0 \\ \frac{I-\bar{E}}{\beta(1-b)+\gamma} \\ (1+\tau_1)\bar{c}_1 - F_K(\bar{k},1)\bar{k} \\ F_L(k^{t-1},1) - (1+\tau_0)\bar{c}_0 - \bar{k} - \frac{I-\bar{E}}{\beta(1-b)+\gamma} \\ (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_1^{t-1} + (1+\tau_0)\bar{c}_0 + \frac{I-\bar{E}}{\beta(1-b)+\gamma}) + \gamma m^{t-1} \\ \bar{E} \end{pmatrix}$$

so that under the policy $(\tau_0, \tau_1, T_0^t, T_1^t)$ agent t's chooses ice the planner's steady state capital ratio $\bar{k}$ and consumption when old $\bar{c}_1$. ■

The next proposition shows that there exist lump-sum transfers T_0^{t+1}, T_1^{t+1} that make agent $t+1$ choose, under the same tax rates τ_0, τ_1, the planner's steady state $\left\{\bar{c}_0, \bar{c}_{0_1}, \bar{k}, \bar{m}, \bar{E}, \bar{E}\right\}$.

Proposition 3: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production the policy* $(\tau_0, \tau_1, T_0, T_1)$ *such that*

$$\tau_0 = \frac{[\gamma+\beta(1+R)](1+R) - [\gamma+\beta(1-b)](b+R)}{(b+R)\gamma} > 0 \tag{51}$$

$$\tau_1 = \frac{1+R}{b+R} \cdot \frac{\gamma+\beta(1-b)}{\gamma-\alpha(1+R)} - 1 > 0 \tag{52}$$

$$T_0 = F_L(\bar{k},1) - (1+\bar{\tau}_0)\bar{c}_0 - \bar{k} - \bar{m} \tag{53}$$

$$T_1 = (1+\tau_1)\bar{c}_1 - F_K(\bar{k},1)\bar{k} \tag{54}$$

implements at $t+1$ the planner's steady state, after the first stage of taxation in period t, and keeps the government budget balanced.

Proof: Given E_t, $c_1^t = \bar{c}_1$, $k^t = \bar{k}$, m^t, $c_0^{t+2} = c_0^{t+2,e}$ and the planner's steady state capital ratio, $\bar{k}$, the planner's steady state consumption of the old, $\bar{c}_1$, and the consumption tax rates τ_0, τ_1, then at the perfect foresight equilibrium, $\left\{c_0^{t+1}, T_0^{t+1}, T_1^{t+1}, m^{t+1}, E_{t+1}, E_{t+2}\right\}$ is characterized by the following system of equation

Agent $t+1$ will also choose the planner's capital ratio and consumption when old under a policy $(\tau_0, \tau_1, T_0^{t+1}, T_1^{t+1})$ -with the same tax rates as before- if

$$u'(c_0^{t+1}) = (1+R)\frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+2}^e) \tag{55}$$

$$v'(\bar{c}_1) = \frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+2}^e) \tag{56}$$

$$(1+\tau_0)c_0^{t+1} + \bar{k} + m^{t+1} = F_L(\bar{k},1) - T_0^{t+1} \tag{57}$$

$$(1+\tau_1)\bar{c}_1 = F_K(\bar{k},1)\bar{k} + T_1^{t+1} \tag{58}$$

$$E_{t+1}(=\bar{E}) = (1-b)E_t - \alpha F(\bar{k},1) - \beta(c_0^{t+1} + \bar{c}_1) + \gamma m^t \tag{59}$$

$$E_{t+2}^e = (1-b)\bar{E} - \alpha F(\bar{k},1) - \beta(c_0^{t+2} + \bar{c}_1) + \gamma m^{t+1} \tag{60}$$

The solution in $\left\{c_0^{t+1}, T_0^{t+1}, T_1^{t+1}, m^{t+1}, E_{t+1}, E_{t+2}^e\right\}$ to this system is the following. Given that $\phi'' < 0$, from (56) and (23) it follows

$$\left.\begin{array}{c} v'(\bar{c}_1) = \frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+2}^e) \\ v'(\bar{c}_1) = \frac{\gamma+\beta(1+R)}{b+R}\phi'(\bar{E}) \end{array}\right. \Rightarrow E_{t+2}^e = \bar{E}$$

From (55), (22) and $E_{t+2} = \bar{E}$

$$\left.\begin{array}{c} u'(c_0^{t+1}) = (1+R)\frac{\gamma+\beta(1+R)}{b+R}\phi'(E_{t+2}^e) \\ u'(\bar{c}_0) = (1+R)\frac{\gamma+\beta(1+R)}{b+R}\phi'(\bar{E}) \\ E_{t+2}^e = \bar{E} \end{array}\right. \Rightarrow c_0^{t+1} = \bar{c}_0$$

At a perfect foresight equilibrium, $c_0^{t+2,e} = c_0^{t+2} = \bar{c}_0$ holds. Substituting $c_0^{t+2} = \bar{c}_0$ and $E_{t+2} = \bar{E}$ into equation (60) and comparing with equation (26) we have

$$\begin{cases} b\bar{E} + \alpha F(\bar{k}, 1) + \beta(\bar{c}_0 + \bar{c}_1) - \gamma m^{t+1} = 0 \\ b\bar{E} + \alpha F(\bar{k}, 1) + \beta(\bar{c}_0 + \bar{c}_1) - \gamma \bar{m} = 0 \end{cases} \Rightarrow m^{t+1} = \bar{m}$$

Substituting $c_0^{t+1} = \bar{c}_0$ and $m^{t+1} = \bar{m}$ into (57) we have

$$T_0^{t+1} = F_L(\bar{k}, 1) - (1 + \tau_0)\bar{c}_0 - \bar{k} - \bar{m} = T_0$$

Equation (58) gives us

$$T_1^{t+1} = (1 + \tau_1)\bar{c}_1 - F_K(\bar{k}, 1)\bar{k} = T_1$$

So, by implementing the taxes and transfers policy, $(\tau_0, \tau_1, T_0, T_1)$, the optimal choice of agent born at date $t+1$ will coincide with the planner's optimal steady state.

We know the old agent in period $t + 1$ also receives a transfer $T_1 = (1 + \tau_1)\bar{c}_1 - F_K(\bar{k}, 1)\bar{k}$. It is obvious that

$$T_1 = \tau_0\bar{c}_0 + \tau_1\bar{c}_1 + T_0$$

So, at such steady state, under this taxes and transfers policy, the government budget is kept balanced every period. ■

5.2. Taxes on consumptions and capital income

In the section 5.1, we introduced taxes on consumptions in which the tax rates differ between consumptions of the old and the young. In the reality, however, this tax scheme seems to be difficult to apply because it may violate the equity among generations. In order to avoid the discrimination between the old and the young, a unique rate of consumption tax τ^t should be applied. Beside that, a capital income tax τ_k^t and a system of lump-sum tax T_0^t (if positive) and lump-sum transfer T_1^t (if positive), levied on agent t's incomes, are introduced to show that the best steady state allocation can be achieved. The problem of agent t is then

$$\underset{\substack{c_0^t, c_1^t, k^t, m^t \geq 0 \\ E_t, E_{t+1}^e}}{Max} u(c_0^t) + v(c_1^t) + \phi(E_{t+1}^e) \tag{61}$$

subject to

$$(1+\tau^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{62}$$

$$(1+\tau^t)c_1^t = (1-\tau_k^t)r_{t+1}k^t + T_1^t \tag{63}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1}, 1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{64}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1}, 1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{65}$$

given E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, w_t, $c_0^{t+1,e}$ and r_{t+1}. Note again that in equation (65), the agent, being negligible within his generation, ignores the fact that $K_{t+1} = k^t$ and hence is unable to internalize the effect of the savings decisions on environment through the aggregate output. Hence, the first-order conditions characterizing agent t's optimal choice are

$$u'(c_0^t) = \left[\beta(1-b) + \gamma(1+\tau^t)\right]\phi'(E_{t+1}^e) \tag{66}$$

$$v'(c_1^t) = \left[\beta + \frac{\gamma(1+\tau^t)}{(1-\tau_k^t)F_K(k^t, 1)}\right]\phi'(E_{t+1}^e) \tag{67}$$

$$(1+\tau^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{68}$$

$$(1+\tau^t)c_1^t = (1-\tau_k^t)r_{t+1}k^t + T_1^t \tag{69}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1}, 1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{70}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1}, 1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{71}$$

At a perfect foresight equilibrium the wage rate and capital return are given by the labor and capital productivities respectively, and forecasts coincide with actual values, i.e. $E_{t+1}^e = E_{t+1}$ and $c_0^{t+1,e} = c_0^{t+1}$.

Proposition 4: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, in any period t, there exists a period by period budget balanced policy of consumption tax, capital income tax and lump-sum taxes and transfers, $(\tau^t, \tau_k^t, T_0^t, T_1^t)$, that supports to attain the planner's steady state capital (saving) ratio, $\bar{k}$, and the planner's steady state consumption of the old, $\bar{c}_1$, at the competitive equilibrium.*

Proof: The proof for this proposition is similar to the proof for proposition 2. ■

Proposition 5: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, after finishing period t (the first stage of taxation), the economy can achieve the planner's steady state from period $t+1$ onward by implementing the following combination*

$$\tau = \frac{[\gamma + \beta(1+R)](1+R) - [\gamma + \beta(1-b)](b+R)}{(b+R)\gamma} > 0$$

$$\tau_k = 1 - \frac{(b+R)(\gamma - (1+R)\alpha)(1+\tau)}{(1+R)(\gamma + \beta - \beta b)}$$

$$T_0 = F_L(\bar{k}, 1) - (1+\tau)\bar{c}_0 - \bar{k} - \bar{m}$$

$$T_1 = (1+\tau)\bar{c}_1 - (1-\tau_k)F_K(\bar{k}, 1)\bar{k}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: The proof for this proposition is similar to the proof for proposition 3. ■

5.3. Taxes on consumptions and production

We still keep the non-discriminatory tax rate τ^t on consumptions and the system of lump-sum tax T_0^t (if positive) and lump-sum transfer T_1^t (if positive). However, we now introduce a Pigouvian tax on production instead of tax on capital income. In any period, let τ_p be the tax paid by firms per one unit of output produced. We will design taxes and transfers policy ensuring the government's budget to be balanced and achieving the planner's steady state through competitive markets.

Under the production tax, the problem that the firm much solve is

$$\underset{K_t}{Max} \quad (1-\tau_p)F(K_t,1) - r_t k_t - w_t \tag{72}$$

The return of capital and the return of labor at the equilibrium are

$$r_t = (1-\tau_p)F_K(k^{t-1},1) \tag{73}$$

$$w_t = (1-\tau_p)F_L(k^{t-1},1) \tag{74}$$

Under the taxes and transfers policy, the agent t's problem is

$$\underset{\substack{c_0^t, c_1^t, k^t, m^t \geq 0 \\ E_t,\, E_{t+1}}}{Max} u(c_0^t) + v(c_1^t) + \phi(E_{t+1}^e) \tag{75}$$

subject to

$$(1+\tau^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{76}$$

$$(1+\tau^t)c_1^t = r_{t+1}k^t + T_1^t \tag{77}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k_t,1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{78}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{79}$$

given E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, w_t, $c_0^{t+1,e}$ and r_{t+1}. Note again that in equation (79), the agent, being negligible within his generation, ignores the fact that $K_{t+1} = k^t$ and hence is unable to internalize the effect of the savings decisions on environment through the aggregate output. Hence, the first-order conditions characterizing agent t's optimal choice are

$$u'(c_0^t) = \left[\beta(1-b) + \gamma(1+\tau^t)\right] \phi'(E_{t+1}^e) \tag{80}$$

$$v'(c_1^t) = \left[\beta + \frac{\gamma(1+\tau^t)}{(1-\tau_p^t)F_K(k^t,1)}\right] \phi'(E_{t+1}^e) \tag{81}$$

$$(1+\tau^t)c_0^t + k^t + m^t = w_t - T_0^t \tag{82}$$

$$(1+\tau^t)c_1^t = (1-\tau_p^t)r_{t+1}k^t + T_1^t \tag{83}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{84}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{85}$$

At a perfect foresight equilibrium the wage rate and capital return are given by the labor and capital productivities respectively, and forecasts coincide with actual values, i.e. $E_{t+1}^e = E_{t+1}$ and $c_0^{t+1,e} = c_0^{t+1}$.

Proposition 6: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, in any period t, there exists a period by period budget balanced policy of consumption tax, production tax, lump-sum taxes and transfers, $(\tau^t, \tau_p^t, T_0^t, T_1^t)$, that supports to attain the planner's steady state capital (saving) ratio, $\bar{k}$, and the planner's steady state consumption of the old, $\bar{c}_1$, at the competitive equilibrium.*

Proof: The proof for this proposition is similar to the proof for proposition 2. ■

Proposition 7: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, after finishing period t (the first stage of taxation), the economy can achieve the planner's steady state from period $t+1$ onward by implementing the following combination*

$$\tau = \frac{[\gamma + \beta(1+R)](1+R) - [\gamma + \beta(1-b)](b+R)}{(b+R)\gamma} > 0$$

$$\tau_p = 1 - \frac{(b+R)(\gamma-(1+R)\alpha)(1+\tau)}{(1+R)(\gamma+\beta-\beta b)}$$

$$T_0 = F_L(\bar{k},1) - (1+\tau)\bar{c}_0 - \bar{k} - \bar{m}$$

$$T_1 = (1+\tau)\bar{c}_1 - (1-\tau_p)F_K(\bar{k},1)\bar{k}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: The proof for this proposition is similar to the proof for proposition 3. ■

5.4. Taxes on consumption, production and labor income

We now modify the tax and transfer policy introduced in section 5.3 by using the labor income tax rate τ_w^t to replace the lump-sum tax T_0^t on wage. All other things are kept the same as in the section 5.3. Under this policy, the agent t's problem is

$$\underset{\substack{c_0^t, c_1^t, k^t, m^t \geq 0 \\ E_t,\, E_{t+1}}}{Max} u(c_t^t) + v(c_{t+1}^t) + \phi(E_{t+1}) \tag{86}$$

subject to

$$(1+\tau_c)c_0^t + k^t + m^t = (1-\tau_w^t)w_t \tag{87}$$

$$(1+\tau)c_1^t = r_{t+1}k^t + T_0^t \tag{88}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{89}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{90}$$

given E_{t-1}, c_1^{t-1}, k^{t-1}, m^{t-1}, w_t, $c_0^{t+1,e}$ and r_{t+1}. Note again that in equation (90), the agent, being negligible within his generation, ignores the fact that $K_{t+1} = k^t$ and hence is unable to internalize the effect of the savings decisions on environment through the aggregate output. Hence, the first-order conditions characterizing agent t's optimal choice are

$$u'(c_0^t) = \left[\beta(1-b) + \gamma(1+\tau^t)\right] \phi'(E_{t+1}^e) \tag{91}$$

$$v'(c_1^t) = \left[\beta + \frac{\gamma(1+\tau^t)}{(1-\tau_p^t)F_K(k^t,1)}\right] \phi'(E_{t+1}^e) \tag{92}$$

$$(1+\tau^t)c_0^t + k^t + m^t = (1-\tau_w^t)w_t \tag{93}$$

$$(1+\tau^t)c_1^t = (1-\tau_p^t)r_{t+1}k^t + T_1^t \tag{94}$$

$$E_t = (1-b)E_{t-1} - \alpha F(k^{t-1},1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1} \tag{95}$$

$$E_{t+1}^e = (1-b)E_t - \alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma m^t \tag{96}$$

At a perfect foresight equilibrium the wage rate and capital return are given by the labor and capital productivities respectively, and forecasts coincide with actual values, i.e. $E_{t+1}^e = E_{t+1}$ and $c_0^{t+1,e} = c_0^{t+1}$.

Proposition 8: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, in any period t, there exists a period by period budget balanced policy of consumption tax, production tax, lump-sum taxes and transfers, $(\tau^t, \tau_p^t, \tau_w^t, T_1^t)$, that supports to attain the planner's steady state capital (saving) ratio, $\bar{k}$, and the planner's steady state consumption of the old, $\bar{c}_1$, at the competitive equilibrium.*

Proof: The proof for this proposition is similar to the proof for proposition 2. ■

Proposition 9: *In a Diamond (1965) overlapping generations economy with pollution from both consumption and production, after finishing period t (the first stage of taxation), the economy can* *achieve the planner's steady state from period $t+1$ onward by implementing the following combination*

$$\tau = \frac{\beta + (1-b)(\gamma - \beta b) + \beta R(1+b+R)}{(b+R)\gamma}$$

$$\tau_p = 1 - \frac{(b+R)(\gamma - (1+R)\alpha)(1+\tau)}{(1+R)(\gamma + \beta - \beta b)}$$

$$\tau_w = 1 - \frac{(1+\tau)\bar{c}_0 + \bar{k} + \bar{m}}{(1-\tau_p)F_L(\bar{k}, 1)}$$

$$T_1 = (1+\tau)\bar{c}_1 - (1-\tau_p)F_K(\bar{k}, 1)\bar{k}$$

At such the steady state the goverment's budget is kept balanced every period.

Proof: The proof for this proposition is similar to the proof for proposition 3. ■

6 Conclusion

We have presented a general equilibrium overlapping generations model with environmental externalities from both production and consumption. For such a model we proved that the competitive equilibrium steady state is not the efficient steady state, for any discount rate the social planner may use. The pollution externality from consumption does not affect the range of dynamically inefficient capital ratios, whereas the pollution externality from production does. The higher the production pollution parameter α, the larger the inefficient range. The environmental maintaining technology γ also plays a role in determining the best steady state capital ratio $\bar{k}$. The cleaner the environment maintaining technology, the smaller the range of the dynamically inefficient allocations. By comparing the competitive steady state and the best steady state, we designed a balanced budget taxes and transfer policy that decentralizes the planner's steady state.

This paper makes many simplifying assumptions, such as the technology being exogenous, the population growth rate being zero and there being only one production sector. Further developments including endogenous technology and fertility, as well as the impact of human capital accumulation are left for future research.

Appendix

A1. Existence of the agent's optimal solution

By substituting (10), (11), (12) and (13) into (8) and (9) the existence of solution to the system of the first order conditions (8)-(13) is equivalent to the existence of solution to the system of two following equations

$$u'(c_0^t) - [\beta(1-b)+\gamma]\,\phi'(E_{t+1}^e) = 0 \tag{97}$$

$$v'(c_1^t) - \frac{\beta + \frac{\gamma}{r_{t+1}}}{\beta(1-b)+\gamma} u'(c_0^t) = 0 \tag{98}$$

where

$$\begin{aligned} E_{t+1}^e = (1-b)\left[(1-b)E_{t-1} - \alpha F(k_{t-1},1) - \beta(c_0^t + c_1^{t-1}) + \gamma m^{t-1}\right] \\ -\alpha F(K_{t+1},1) - \beta(c_0^{t+1,e} + c_1^t) + \gamma(w_t - c_0^t - \tfrac{c_1^t}{r_{t+1}}) \end{aligned}$$

From (98), by implicit function theorem we can treat c_1^t as a function of c_0^t, $c_1^t = \varphi(c_0^t)$ where $\varphi'(\cdot) > 0$, $\varphi(0) = 0$ and $\varphi(+\infty) = +\infty$. We rewrite

$$E_{t+1}^e = Q - [\beta(1-b)+\gamma]\,c_0^t - (\beta + \frac{\gamma}{r_{t+1}})\varphi(c_0^t)$$

where $Q = (1-b)\left[(1-b)E_{t-1} - \alpha F(k_{t-1},1) - \beta c_1^{t-1} + \gamma m^{t-1}\right] - \alpha F(K_{t+1},1) - \beta c_0^{t+1,e} + \gamma w_t$. Now the system of equations (97) and (98) leads to the following equation

$$u'(c_0^t) - [\beta(1-b)+\gamma]\,\phi'(Q - [\beta(1-b)+\gamma]\,c_0^t - (\beta + \frac{\gamma}{r_{t+1}})\varphi(c_0^t)) = 0 \tag{99}$$

The existence of the agent's optimal solution is equivalent to the existence of solution to equation (99). In effect, set

$$\triangle = u'(c_0^t) - [\beta(1-b)+\gamma]\,\phi'(Q - [\beta(1-b)+\gamma]\,c_0^t - (\beta + \frac{\gamma}{r_{t+1}})\varphi(c_0^t))$$

is a continuous function of c_0^t. We have,

$$\lim_{c_0^t \to +\infty} \triangle = -\infty < 0$$

and

$$\lim_{c_0^t \to 0^+} \triangle = +\infty > 0$$

We also find that $\triangle$ is a monotone function of c_0^t since

$$\frac{\partial \triangle}{\partial c_0^t} = u''(c_0^t) + [\beta(1-b)+\gamma] \left[\beta(1-b) + \gamma + (\beta + \frac{\gamma}{r_{t+1}})\varphi'(c_0^t)\right] \phi''(E_{t+1}^e) < 0$$

So there exists a unique solution $c_0^t > 0$ to (99), meaning that there exists a unique optimal solution of the agent.

A2. Competitive equilibrium dynamics

The competitive equilibrium conditions impose on $(c_0^{t+1}, c_1^t, k^t, m^t, E_{t+1})$ a dynamics described by a first-order difference equation, since

$$u'(c_0^t) - [\beta(1-b)+\gamma]\, \phi'(E_{t+1}) = 0$$

$$v'(c_1^t) - \left[\beta + \frac{\gamma}{F_K(k^t,1)}\right] \phi'(E_{t+1}) = 0$$

$$c_0^t + k^t + m^t - F_L(k^{t-1},1) = 0$$

$$c_1^t - F_K(k^t,1)k^t = 0$$

$$E_{t+1} - (1-b)E_t + \alpha F(k^t,1) + \beta(c_0^{t+1} + c_1^t) - \gamma m^t = 0$$

implicitly define it to be a function of its lagged value $(c_0^t, c_1^{t-1}, k^{t-1}, m^{t-1}, E_t)$. In effect, the associated Jacobian matrix with respect to $(c_0^{t+1}, c_1^t, k^t, m^t, E_{t+1})$

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & G \\ 0 & v''(c_1^t) & D & 0 & H \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -C & 0 & 0 \\ \beta & \beta & \alpha F_K(k^t,1) & -\gamma & 1 \end{pmatrix}$$

where

$$C = F_K(k^t, 1) + F_{KK}(k^t, 1)k^t = \theta^2 A(k^t)^{\theta-1} > 0$$

$$D = \frac{\gamma F_{KK}(k^t, 1)}{F_K(k^t, 1)^2}\phi'(E_{t+1}) < 0$$

$$G = -\left[\beta(1-b) + \gamma\right]\phi''(E_{t+1}) > 0$$

$$H = -\left[\beta + \frac{\gamma}{F_K(k^t, 1)}\right]\phi''(E_{t+1}) > 0$$

is regular, since

$$\begin{aligned}
det(J) = G &\begin{vmatrix} 0 & v''(c_1^t) & D & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -C & 0 \\ \beta & \beta & \alpha F_K(k^t, 1) & -\gamma \end{vmatrix} \\
= -G\beta &\begin{vmatrix} v''(c_1^t) & D & 0 \\ 0 & 1 & 1 \\ 1 & -C & 0 \end{vmatrix} \\
= -G\beta &\left(D + Cv''(c_1^t)\right) > 0
\end{aligned}$$

Since, the Jacobian matrix is regular for all $(c_0^{t+1}, c_1^t, k^t, m^t, E_{t+1})$ then it is evidently regular at the solution. This implies that for all competitive equilibrium $(c_0^{t+1}, c_1^t, k^t, m^t, E_{t+1})_t$ there exists, for all t, a function $\psi : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\begin{pmatrix} c_0^{t+1} \\ c_1^t \\ k^t \\ m^t \\ E_{t+1} \end{pmatrix} = \psi \begin{pmatrix} c_0^t \\ c_1^{t-1} \\ k^{t-1} \\ m^{t-1} \\ E_t \end{pmatrix}$$

A3. Checking the SOCs for the maximization problem of the agent

For the FOCs to be sufficient conditions to characterize a (local) maximum to the optimization problem, we have to check the sufficient SOCs. The Lagrangian of the maximization problem is

$$Z = u(c_0^t) + v(c_1^t) + \phi(E_{t+1}) + \lambda_1(c_0^t + k^t + m^t - w_t) + \lambda_2(c_1^t - r_{t+1}k^t)$$

$$+\lambda_3\left(E_t - (1-b)E_{t-1} + \alpha F(k^{t-1},1) + \beta(c_0^t + c_1^{t-1}) - \gamma m^{t-1}\right)$$

$$+\lambda_4\left(E_{t+1} - (1-b)E_t + \alpha F(k^t,1) + \beta(c_0^{t+1,e} + c_1^t) - \gamma m_t\right)$$

whose bordered Hessian will appear as

$$\bar{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 & 1 \\ 1 & 0 & \beta & 0 & u''(c_0^t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_1^t) & 0 & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi''(E_{t+1}) \end{pmatrix}$$

The sufficient SOCs for a maximum are

$$(-1)^5\left|\bar{H}_5\right| = -\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 \\ 1 & 0 & \beta & 0 & u''(c_0^t) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_1^t) & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= -\left(\beta(1-b)+\gamma\right)^2 r_{t+1}^2 v''(c_1^t) - \left(\beta r_{t+1}+\gamma\right)^2 u''(c_0^t) > 0$$

$$(-1)^6 \left|\bar{H}_6\right| = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -r_{t+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & -\gamma & b-1 & 1 \\ 1 & 0 & \beta & 0 & u''(c_0^t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \beta & 0 & v''(c_1^t) & 0 & 0 & 0 & 0 \\ 1 & -r_{t+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \phi''(E_{t+1}) \end{vmatrix}$$

$$= r_{t+1}^2 v''(c_1^t) u''(c_0^t) + \phi''(E_{t+1}) \left|\bar{H}_5\right| > 0$$

which guarantees that the solution to the agent's problem is a maximum indeed.

A4. Solving the problem of the social planner

The Lagrange function for this problem is

$$\mathcal{L} = \sum_{t=0}^{+\infty} \frac{1}{(1+R)^t} \left[u(c_0^t) + u(c_1^t) + \phi(E_{t+1})\right]$$

$$+ \sum_{t=0}^{+\infty} \frac{\mu_t}{(1+R)^t} \left[F(k^{t-1}, 1) - c_0^t - c_1^{t-1} - k^t - m^t\right]$$

$$+ \sum_{t=0}^{+\infty} \frac{\eta_t}{(1+R)^t} \left[E_{t+1} - (1-b)E_t + \alpha F(k^t, 1) + \beta(c_0^{t+1} + c_1^t) - \gamma m^t\right]$$

The FOCs of the maximization problem are

$$\frac{\partial \mathcal{L}}{\partial c_t^t} = \frac{u'(c_0^t)}{(1+R)^t} - \frac{\mu_t}{(1+R)^t} + \frac{\beta \eta_{t-1}}{(1+R)^{t-1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial c_1^t} = \frac{v'(c_1^t)}{(1+R)^t} - \frac{\mu_{t+1}}{(1+R)^{t+1}} + \frac{\beta \eta_t}{(1+R)^t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial E_{t+1}} = \frac{\phi'(E_{t+1})}{(1+R)^t} + \frac{\eta_t}{(1+R)^t} - \frac{\eta_{t+1}(1-b)}{(1+R)^{t+1}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial k^t} = -\frac{\mu_t}{(1+R)^t} + \frac{\mu_{t+1} F_K(k^t, 1)}{(1+R)^{t+1}} + \frac{\eta_t \alpha F_K(k^t, 1)}{(1+R)^t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial m^t} = -\frac{\mu_t}{(1+R)^t} - \frac{\eta_t \gamma}{(1+R)^t} = 0$$

that is to say

$$u'(c_0^t) - \mu_t + \frac{\beta \eta_{t-1}}{1+R} = 0$$

$$v'(c_1^t) - \frac{\mu_{t+1}}{1+R} + \beta \eta_t = 0$$

$$\phi'(E_{t+1}) + \eta_t - \frac{\eta_{t+1}(1-b)}{1+R} = 0$$

$$-\mu_t + \frac{\mu_{t+1} F_K(k^t, 1)}{1+R} + \eta_t \alpha F_K(k^t, 1) = 0$$

$$-\mu_t - \eta_t \gamma = 0$$

At the steady state,

$$u'(\bar{c}_0) = \mu - \beta \eta (1+R)$$

$$v'(\bar{c}_1) = \frac{\mu}{1+R} - \beta \eta$$

$$\phi'(\bar{E}) = -\eta + \frac{(1-b)\eta}{1+R}$$

$$F_K(\bar{k}, 1) = \frac{\mu(1+R)}{\mu + \alpha \eta (1+R)}$$

$$\mu = -\eta \gamma$$

Therefore,

$$u'(\bar{c}_0) = (1+R)\frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E})$$

$$v'(\bar{c}_1) = \frac{\gamma + \beta(1+R)}{b+R}\phi'(\bar{E})$$

$$F_K(\bar{k}, 1) = \frac{\gamma(1+R)}{\gamma - (1+R)\alpha}$$

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